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## On the deficiencies of meromorphic functions of finite order

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# ON THE DEFICIENCIES OF MEROMORPHIC FUNCTIONS OF FINITE ORDER

MASAYOSHI FURUTA

## 1. Introduction

Let  $f(z)$  be a meromorphic function in  $|z| < \infty$ . The standard symbols of the Nevanlinna theory

$$\log^+, m(r, a), N(r, a), T(r, f), z = re^{i\theta},$$

are used throughout this note. Moreover, we use the following notations.

The order  $\lambda$  and lower order  $\mu$  of  $f(z)$  are defined by

$$\lambda = \limsup_{r \rightarrow \infty} \frac{\log T(r, f)}{\log r}, \quad \mu = \liminf_{r \rightarrow \infty} \frac{\log T(r, f)}{\log r}.$$

The deficiencies  $\delta(a, f)$  in the sense of Nevanlinna and  $\Delta(a, f)$  in the sense of Valiron, of the value  $a$ , are defined by, respectively:

$$\delta(a, f) = \liminf_{r \rightarrow \infty} \frac{m(r, a)}{T(r, f)}, \quad \Delta(a, f) = \limsup_{r \rightarrow \infty} \frac{m(r, a)}{T(r, f)},$$

where the total deficiency  $\sum_a \delta(a, f) \leq 2$ .

The quantity  $\kappa(f)$  is defined by

$$\kappa(f) = \limsup_{r \rightarrow \infty} \frac{N(r, 0) + N(r, \infty)}{T(r, f)} \leq 2 - \delta(0, f) - \delta(\infty, f).$$

Now, one of the conjectures of F. Nevanlinna [5] is that  $\delta(a_k, f) = q(k)/\lambda$  ( $q(k)$ : an integer) if  $f(z)$  is a meromorphic function of finite order  $\lambda$  with  $\sum_a \delta(a, f) = 2$ .

A. Pfluger [6] showed that the conjecture is valid for the entire functions, and A. Edrei [2, p. 54] pointed out without proof that it is also valid for meromorphic functions with  $\sum_{a \neq \infty} \delta(a, f) = 1$  and  $\delta(\infty, f) = 1$ . We shall show that  $q(k)$  is the number of asymptotic paths with the asymptotic value  $a_k$  (Theorem 1).

It is plausible that contributions, to a deficiency  $\delta(a_k, f)$ , of  $f$  near the paths are even, and considering a subdivision of deficiencies, we

obtain some results that reflect this in a sense (Theorem 2 and its Corollaries).

The method of proofs in this note is mainly due to A. Edrei and W. H. J. Fuchs [4].

## 2. Statement and discussion of results.

We take  $a_0=0$ ,  $a_\infty=\infty$  and finite values  $a_k$  ( $k=1, 2, \dots, s$ ).

Let  $f(z)$  be a meromorphic function of finite order  $\lambda$  in  $|z|<\infty$ , and  $p$  the integer defined by

$$p - \frac{1}{2} \leq \lambda < p + \frac{1}{2}.$$

From Theorem 6 of [3] (p. 298), Lemma 1 of [3] (pp. 298–299) and Theorem 3 of [1] (p. 173), we see that if

$$(2.1) \quad \sum_{a \neq \infty} \delta(a, f) = 1 \quad \text{and} \quad \delta(\infty, f) = 1,$$

then

$$(2.2) \quad \lambda = \mu \quad \text{and} \quad \lambda = p \geq 1.$$

Frequently we use the following

**Lemma A** (Edrei and Fuchs [4], p. 279). *Let  $f(z)$  be a meromorphic function of finite order  $\lambda$  in  $|z|<\infty$ . Give  $\varepsilon$  ( $0<\varepsilon<1/16$ ) and  $\delta$  ( $0<\delta<1/e$ ) arbitrarily. If  $\lambda(f)=0$ , then  $\lambda=p$  and there exists a sequence  $\{c_n\}$  ( $c_n=c(\alpha^n)$ ;  $\alpha=e^{\frac{1}{p+1}}$ ) such that*

*$|\log|f(z)| - \operatorname{Re} c_n z^n| < 4\varepsilon |c_n| r^n$  for  $z \in \Gamma_n - E_n$  ( $|z|=r$ ) if  $r$  is sufficiently large,  $r > r_0$ , where*

$$\Gamma_n = \{z; \alpha^n \leq |z| < \alpha^{n+\frac{3}{2}}\}$$

*and the exceptional set  $E_n$  is contained in the finite number of discs in  $\Gamma_{n-1} \cup \Gamma_n \cup \Gamma_{n+1}$ , the sum of whose radii doesn't exceed  $4e^2 \delta \alpha^n$ .*

From Theorems 2 and 3 of [4] (pp. 263–264), we see that if (2.1) holds, there exist  $p$  asymptotic paths  $\mathcal{L}^{(k)}$  ( $k=1, 2, \dots, p$ ) with finite asymptotic values such that each of  $\mathcal{L}^{(k)}$  ( $k=2, \dots, p$ ) is the rotation of  $\mathcal{L}^{(1)}$ , around the origin, of angle  $(k-1)2\pi/p$ , and denoting by  $a_k$  ( $k=1, 2, \dots, s$ ;  $s \leq p$ ) distinct values among the asymptotic values corresponding to these paths, then

$$(2.3) \quad \sum_{k=1}^s \delta(a_k, f) = 1.$$

As an immediate consequence of this result, we obtain the following

**Theorem 1** (see. Edrei [2], p. 54). *Let  $f(z)$  be a meromorphic function of finite order  $\lambda$  in  $|z| < \infty$ . If (2.1) holds, then*

$$\delta(a_k, f) = q(k)/\lambda,$$

where  $q(k)$  is the number of asymptotic paths with the asymptotic value  $a_k$  ( $k=1, 2, \dots, s$ ).

Now, we consider a subdivision of deficiencies. To do this we need the following definitions;

$$D_k = \left\{ z; \frac{1}{|f(z) - a_k|} > 1 \right\} \cap \{z; |z| > r_0\},$$

$$D_\infty = \{z; |f(z)| > 1\} \cap \{z; |z| > r_0\} \quad (r_0 > 0)$$

and  $D_k^{(l)}$  ( $l=1, 2, \dots, q(k)$ ) or  $D_\infty^{(l)}$  ( $l=1, 2, \dots, q(\infty)$ ) are the disjoint components of  $D_k$  or  $D_\infty$ , respectively. We set

$$\begin{aligned} F_k(r) &= \{\theta; z = re^{i\theta} \in D_k\} & \text{where } F_k(r) \subset [0, 2\pi), \\ F_\infty(r) &= \{\theta; z = re^{i\theta} \in D_\infty\} & \text{where } F_\infty(r) \subset [0, 2\pi), \\ F_k^{(l)}(r) &= \{\theta; z = re^{i\theta} \in D_k^{(l)}\} & (l=1, 2, \dots, q(k)), \\ F_\infty^{(l)}(r) &= \{\theta; z = re^{i\theta} \in D_\infty^{(l)}\} & (l=1, 2, \dots, q(\infty)). \end{aligned}$$

Further we set

$$m(r, a_k, f; F_k^{(l)}(r)) = \frac{1}{2\pi} \int_{F_k^{(l)}(r)} \log^+ \frac{1}{|f(z) - a_k|} d\theta \quad (l=1, 2, \dots, q(k))$$

$$m(r, \infty, f; F_\infty^{(l)}(r)) = \frac{1}{2\pi} \int_{F_\infty^{(l)}(r)} \log^+ |f(z)| d\theta \quad (l=1, 2, \dots, q(\infty))$$

and

$$\delta^{(l)}(a_k, f) = \liminf_{r \rightarrow \infty} \frac{m(r, a_k, f; F_k^{(l)}(r))}{T(r, f)}$$

$$\Delta^{(l)}(a_k, f) = \limsup_{r \rightarrow \infty} \frac{m(r, a_k, f; F_k^{(l)}(r))}{T(r, f)} \quad (l=1, 2, \dots, q(k)).$$

Then we have

$$\sum_l \delta^{(l)}(a_k, f) \leq \delta(a_k, f) \leq \Delta(a_k, f) \leq \sum_l \Delta^{(l)}(a_k, f).$$

When we take  $f'(z)$  for  $f(z)$  in the above definitions, we use the symbols  $G_0(r)$ ,  $G_\infty(r)$ ,  $G_0^{(l)}(r)$  and  $G_\infty^{(l)}(r)$  in place of  $F_0(r)$ ,  $F_\infty(r)$ ,  $F_0^{(l)}(r)$  and  $F_\infty^{(l)}(r)$ , respectively.

We set

$$\begin{aligned} \bar{\delta}(a_k, f) &= \liminf_{r \rightarrow \infty} \frac{m(r, a_k, f; G_\infty(r))}{T(r, f)}, & \underline{\delta}(a_k, f) &= \liminf_{r \rightarrow \infty} \frac{m(r, a_k, f; G_0(r))}{T(r, f)}, \\ \bar{\delta}^{(l)}(a_k, f) &= \liminf_{r \rightarrow \infty} \frac{m(r, a_k, f; G_\infty^{(l)}(r))}{T(r, f)}, & \underline{\delta}^{(l)}(a_k, f) &= \liminf_{r \rightarrow \infty} \frac{m(r, a_k, f; G_0^{(l)}(r))}{T(r, f)}, \\ \bar{\Delta}(a_k, f) &= \limsup_{r \rightarrow \infty} \frac{m(r, a_k, f; G_\infty(r))}{T(r, f)}, & \underline{\Delta}(a_k, f) &= \limsup_{r \rightarrow \infty} \frac{m(r, a_k, f; G_0(r))}{T(r, f)}, \\ \bar{\Delta}^{(l)}(a_k, f) &= \limsup_{r \rightarrow \infty} \frac{m(r, a_k, f; G_\infty^{(l)}(r))}{T(r, f)}, & \underline{\Delta}^{(l)}(a_k, f) &= \limsup_{r \rightarrow \infty} \frac{m(r, a_k, f; G_0^{(l)}(r))}{T(r, f)}. \end{aligned}$$

Using Lemma A, we obtain the following

**Theorem 2.** *Let  $f(z)$  be a meromorphic function of finite order  $\lambda$  in  $|z| < \infty$ . If  $\delta(0, f) = 1$  and  $\delta(\infty, f) = 1$ , then*

$$\begin{aligned} \delta(0, f) &= \sum_{l=1}^{\lambda} \delta^{(l)}(0, f) = 1, & \delta(\infty, f) &= \sum_{l=1}^{\lambda} \delta^{(l)}(\infty, f) = 1, \\ \delta^{(l)}(0, f) &= 1/\lambda, & \delta^{(l)}(\infty, f) &= 1/\lambda \quad (l=1, 2, \dots, \lambda). \end{aligned}$$

**Corollary 1.** *Let  $f(z)$  be an entire function of finite order  $\lambda$ . If  $\delta(0, f) = 1$ , then*

$$\begin{aligned} \Delta(\infty, f) &= \sum_{l=1}^{\lambda} \Delta^{(l)}(\infty, f) = 1, \\ \delta^{(l)}(\infty, f) &= \Delta^{(l)}(\infty, f) = 1/\lambda \quad (l=1, 2, \dots, \lambda). \end{aligned}$$

**Corollary 2.** *Let  $f(z)$  be a meromorphic function of finite order  $\lambda$  in  $|z| < \infty$ . If (2.1) holds, then*

$$\begin{aligned} \delta(\infty, f) &= \bar{\delta}(\infty, f) = \sum_{l=1}^{\lambda} \bar{\delta}^{(l)}(\infty, f) = 1, & \bar{\delta}^{(l)}(\infty, f) &= 1/\lambda \quad (l=1, 2, \dots, \lambda), \\ \bar{\delta}(\infty, f) &= 0 & \text{(so that } \bar{\delta}^{(l)}(\infty, f) &= 0 \quad (l=1, 2, \dots, \lambda)), \\ \bar{\Delta}(a_k, f) &= 0 & \text{(so that } \bar{\Delta}^{(l)}(a_k, f) &= 0 \quad (l=1, 2, \dots, \lambda)), \\ \delta(a_k, f) &= \Delta(a_k, f) = \underline{\Delta}(a_k, f), & \underline{\delta}^{(l)}(a_k, f) &\leq 1/\lambda \quad (l=1, 2, \dots, \lambda). \end{aligned}$$

From Theorem 1 and Corollary 2, the author thinks that the following

could be proved: *Under the condition (2.1),*

$$\delta(\infty, f) = \sum_{l=1}^{\lambda} \delta^{(l)}(\infty, f) = 1, \quad \sum_{k=1}^s \sum_{l=1}^{q(k)} \delta^{(l)}(a_k, f) = 1,$$

$$\delta^{(l)}(\infty, f) = 1/\lambda \quad (l=1, 2, \dots, \lambda), \quad \delta^{(l)}(a_k, f) = 1/\lambda \quad (l=1, 2, \dots, q(k)).$$

Theorem 2 shows that this is valid in the special case that  $\delta(0, f) = 1$  and  $\delta(\infty, f) = 1$ .

### 3. Proof of Theorem 1

We put  $\delta = 1/(p+1)^{11}$ , where  $p = \lambda$  by (2.2), and take  $\varepsilon$  ( $0 < \varepsilon < 1/16$ ) arbitrarily.

Since  $f(z)$  satisfies (2.1),  $\kappa(f') = 0$  by Lemma 1 of [3] (pp. 298—299). Hence, by Lemma A and Theorem 1 of [4] (pp. 261—262), there exist sequences  $\{r_n\}$  and  $\{\tilde{c}_n\}$  such that

$$\alpha^n \leq r_n < \alpha^{n+\frac{1}{2}} \quad (\alpha = e^{\frac{1}{p+1}}), \quad \{z; |z| = r_n\} \cap (\tilde{E}_n \cup \tilde{E}_{n-1}) = \emptyset,$$

$$(3.1) \quad |\log |f'(z)| - \operatorname{Re} \tilde{c}_n z^p| < 4\varepsilon |\tilde{c}_n| r_n^p \quad (|z| = r_n; n > n_0)$$

and

$$(3.2) \quad |\tilde{c}_n| r_n^p = (1 + o(1)) \pi T(r_n, f') \quad \text{on } \Gamma_n, \quad \text{where } \tilde{E}_n \text{ are the exceptional set for } f'(z).$$

Let  $\tilde{\omega}_n$  be the argument of  $\tilde{c}_n$ . We put  $\tilde{G}_0(r_n) = \{\theta; \cos(p\theta + \tilde{\omega}_n) \leq -5\varepsilon\}$ . Then  $\tilde{G}_0(r_n) = \sum_{l=1}^p \tilde{G}_0^{(l)}(r_n)$ , where each  $\tilde{G}_0^{(l)}(r_n)$  is a component of  $\tilde{G}_0(r_n)$  contained in  $\left[\frac{(4l-3)}{2p}\pi - \frac{\tilde{\omega}_n}{p}, \frac{(4l-1)}{2p}\pi - \frac{\tilde{\omega}_n}{p}\right]$  ( $l = 1, 2, \dots, p$ ). We put  $\mathscr{A}_n^{(l)} = \{r_n e^{i\theta}; \theta \in \tilde{G}_0^{(l)}(r_n)\}$  ( $l = 1, 2, \dots, p$ ). Let  $q(1)$  be the number of asymptotic paths with an asymptotic value  $a_1$ . We may assume without loss of generality that  $\mathscr{L}^{(l)}$  ( $l = 1, 2, \dots, q(1)$ ) are these asymptotic paths.

As shown in [4] (pp. 289—290)

$$\lim_{n \rightarrow \infty} f(z) = a_1$$

uniformly on  $\mathscr{A}_n^{(l)}$  for each  $l = 1, 2, \dots, q(1)$ .

$$\text{We put } G_0^l(r_n) = \sum_{l=1}^{q(1)} \tilde{G}_0^{(l)}(r_n). \quad \text{Then}$$

$$\sum_{k=2}^s \frac{1}{2\pi} \int_{c_0^1(r_n)} \log^+ \frac{1}{|f(z) - a_k|} d\theta = O(1) \quad (|z| = r_n; \quad n > n_0).$$

In view of (3.1) and (3.2), we deduce from Lemma 7 of [4] (p. 285)

$$\begin{aligned} \sum_{k=2}^s \frac{1}{2\pi} \int_{\tilde{c}_0(r_n) - c_0^1(r_n)} \log^+ \frac{1}{|f(z) - a_k|} d\theta &\leq \frac{1}{2\pi} \int_{\tilde{c}_0(r_n) - c_0^1(r_n)} \log^+ \frac{1}{|f'(z)|} d\theta + O(\log r_n) \\ &\leq \frac{1}{2\pi} |\tilde{c}_n| r_n^p (p - q(1)) \int_{-\frac{\pi}{2p}}^{\frac{\pi}{2p}} \cos(p\theta) d\theta + 4\epsilon |\tilde{c}_n| r_n^p + O(\log r_n) \\ &= (1 + o(1)) \left(1 - \frac{q(1)}{p}\right) + 4\pi\epsilon T(r_n, f') + O(\log r_n), \\ \sum_{k=2}^s \frac{1}{2\pi} \int_{[0, 2\pi] - \tilde{c}_0(r_n)} \log^+ \frac{1}{|f(z) - a_k|} d\theta &\leq \frac{1}{2\pi} \int_{[0, 2\pi] - \tilde{c}_0(r_n)} \log^+ \frac{1}{|f'(z)|} d\theta + O(\log r_n) \\ &\leq 9(1 + o(1))\pi\epsilon T(r_n, f') + O(\log r_n) \quad (|z| = r_n). \end{aligned}$$

Therefore

$$\sum_{k=2}^s m(r_n, a_k) \leq (1 + o(1)) \left(1 - \frac{q(1)}{p}\right) + 13\pi\epsilon T(r_n, f') + O(\log r_n).$$

By Lemma 1 of [3] (pp. 298—299)

$$(3.3) \quad \lim_{r \rightarrow \infty} \frac{T(r, f')}{T(r, f)} = 1.$$

Hence

$$\sum_{k=2}^s \delta(a_k, f) \leq \sum_{k=2}^s \liminf_{n \rightarrow \infty} \frac{m(r_n, a_k)}{T(r_n, f)} \leq \liminf_{n \rightarrow \infty} \sum_{k=2}^s \frac{m(r_n, a_k)}{T(r_n, f)} \leq 1 - \frac{q(1)}{p} + 13\pi\epsilon.$$

Thus, we deduce from (2.3), being  $\epsilon$  ( $0 < \epsilon < 1/16$ ) arbitrary,

$$\delta(a_1, f) \geq q(1)/p.$$

Similarly, we have for  $2 \leq k \leq s$ ,

$$\delta(a_k, f) \geq q(k)/p.$$

Since  $\sum_{k=1}^s q(k) = p$ , we deduce from (2.3)

$$\delta(a_k, f) = q(k)/p \quad (k = 1, 2, \dots, s).$$

Since  $p = \lambda$ , the proof of Theorem 1 is now complete.

## 4. Proof of Theorem 2

We take  $\varepsilon$  ( $0 < \varepsilon < 1/16$ ) and  $\delta$  ( $0 < \delta < 1/e$ ) arbitrarily. By Lemma A and Theorem 1 of [4] (pp. 261–262), there exist sequences  $\{r_n\}$  and  $\{c_n\}$  such that

$$(4.1) \quad \alpha_n \leq r_n < \alpha_{n+2}, \quad \{z; |z| = r_n\} \cap (E_n \cup E_{n-1}) = \emptyset \\ |\log |f(z)| - \operatorname{Re} c_n z^p| < 4\varepsilon |c_n| r_n^p \quad (|z| = r_n)$$

and

$$(4.2) \quad |c_n| r^p = (1 + o(1)) \pi T(r, f) \quad (|z| = r, z \in \Gamma_n).$$

We put  $c_n = |c_n| e^{i\omega_n}$  ( $c_n \neq 0$ ) and  $\tilde{F}_\infty(r, n) = \{\theta; \cos(p\theta + \omega_n) \geq 4\varepsilon\}$ . Then we have  $\tilde{F}_\infty(r, n) = \sum_{l=1}^p \tilde{F}_\infty^{(l)}(r, n)$ , where each  $\tilde{F}_\infty^{(l)}(r, n)$  is a component of  $\tilde{F}_\infty(r, n)$  contained in  $\left[\frac{(4l-1)\pi - \omega_n}{2p}, \frac{(4l+1)\pi - \omega_n}{2p}\right]$  ( $l=1, 2, \dots, p$ ).

In view of (4.1) and (4.2)

$$\begin{aligned} m(r_n, \infty, f; \tilde{F}_\infty^{(l)}(r_n)) &\leq \frac{1}{2\pi} |c_n| r_n^p \int_{\tilde{F}_\infty^{(l)}(r_n)} \{\cos(p\theta + \omega_n) + 4\varepsilon\} d\theta \\ &\leq \frac{1}{2\pi} |c_n| r_n^p \int_{-\frac{\pi}{2p}}^{\frac{\pi}{2p}} \cos(p\theta) d\theta + 4\varepsilon |c_n| r_n^p \\ &= (1 + o(1)) \left(\frac{1}{p} + 4\pi\varepsilon\right) T(r_n, f), \end{aligned}$$

$$m(r_n, \infty, f; F_\infty^{(l)}(r_n) - \tilde{F}_\infty^{(l)}(r_n)) \leq 8(1 + o(1)) \pi \varepsilon T(r_n, f) \quad (n > n_0),$$

where  $\tilde{F}_\infty^{(l)}(r_n) = \tilde{F}_\infty^{(l)}(r_n, n)$ , so that we have

$$m(r_n, \infty, f; F_\infty^{(l)}(r_n)) \leq (1 + o(1)) \left(\frac{1}{p} + 12\pi\varepsilon\right) T(r_n, f).$$

Hence, we have, being  $\varepsilon$  ( $0 < \varepsilon < 1/16$ ) arbitrary,

$$(4.3) \quad \delta^{(l)}(\infty, f) \leq \liminf_{n \rightarrow \infty} \frac{m(r_n, \infty, f; F_\infty^{(l)}(r_n))}{T(r_n, f)} \leq \frac{1}{p} \quad (l=1, 2, \dots, p).$$

Let  $J(r, n)$  be the part of the exceptional set  $E_n$  on  $|z| = r$ , i. e.,  $J(r, n) = \{\theta; z = re^{i\theta} \in E_n\}$ . Then we have



$$(4.4) \quad \text{meas } J(r, n) \leq 8\pi e^2 \delta.$$

In view of Lemma A, (4.2) and (4.4)

$$\begin{aligned} m(r, \infty, f; F_{\infty}^{(l)}(r)) &= \frac{1}{2\pi} \int_{F_{\infty}^{(l)}(r)} \log^+ |f(z)| d\theta \geq \frac{1}{2\pi} \int_{\bar{F}_{\infty}^{(l)}(r, n) - J(r, n)} \log^+ |f(z)| d\theta \\ &\geq \frac{1}{2\pi} |c_n| r^n \int_{\bar{F}_{\infty}^{(l)}(r, n) - J(r, n)} \{\cos(p\theta + \omega_n) - 4\epsilon\} d\theta \\ &\geq |c_n| r^n \left\{ \frac{1}{2\pi} \int_{-(1-4\epsilon)\frac{\pi}{2p}}^{(1-4\epsilon)\frac{\pi}{2p}} \cos(p\theta) d\theta - \frac{1}{2\pi} \text{meas } J(r, n) - 4\epsilon \right\} \\ &\geq (1+o(1)) \left( \frac{1-4\epsilon}{p} - 4\pi e^2 \delta - 4\pi \epsilon \right) T(r, f), \end{aligned}$$

so that we have

$$(4.5) \quad \delta^{(l)}(\infty, f) \geq \frac{1}{p} - \frac{4\epsilon}{p} - 4\pi \epsilon - 4\pi e^2 \delta.$$

We deduce from (4.3) and (4.5), being  $\epsilon$  ( $0 < \epsilon < 1/16$ ) and  $\delta$  ( $0 < \delta < 1/e$ ) arbitrary,

$$\delta^{(l)}(\infty, f) = 1/p \quad (l=1, 2, \dots, p).$$

Thus, we have

$$\delta(\infty, f) = \sum_{l=1}^p \delta^{(l)}(\infty, f) = 1.$$

Next, taking  $1/f$  for  $f$ , we have by the same calculation as the above one

$$\delta^{(l)}(0, f) = 1/p \quad (l=1, 2, \dots, p).$$

Thus, we have

$$\delta(0, f) = \sum_{l=1}^p \delta^{(l)}(0, f) = 1.$$

Since  $p=\lambda$ , the proof of Theorem 2 is now complete.

## 5. Proofs of corollaries

(1). **Proof of Corollary 1.** By Lemma A for an entire function  $f(z)$ ,

$$(5.1) \quad \log |f(z)| < \operatorname{Re} c_n z^p + 4\varepsilon |c_n| r^p \quad \text{on } \Gamma_n.$$

In view of (4.2) and (5.1)

$$m(r, \infty, f; F_\infty^{(l)}(r)) \leq (1+o(1)) \left( \frac{1}{p} + 12\pi\varepsilon \right) T(r, f),$$

$\varepsilon$  ( $0 < \varepsilon < 1/16$ ) being arbitrary, so that we have,

$$J^{(l)}(\infty, f) \leq 1/p \quad (l=1, 2, \dots, p).$$

As  $\delta^{(l)}(\infty, f) \leq J^{(l)}(\infty, f)$ , we deduce from Theorem 2

$$\delta^{(l)}(\infty, f) = J^{(l)}(\infty, f) = 1/p \quad (l=1, 2, \dots, p)$$

and

$$J(\infty, f) = \sum_{l=1}^p J^{(l)}(\infty, f) = 1.$$

(II). **Proof of Corollary 2.** We deduce from [7] (pp. 23—24), (2.2) and Theorem 2

$$(5.2) \quad \delta^{(l)}(0, f') = 1/p \quad (l=1, 2, \dots, p),$$

$$(5.3) \quad \delta^{(l)}(\infty, f') = 1/p \quad (l=1, 2, \dots, p).$$

Since  $\log^+ |f'| \leq \log^+ |f| + \log^+ \left| \frac{f'}{f} \right|$ ,

$$\begin{aligned} m(r, \infty, f'; G_\infty^{(l)}(r)) &\leq m(r, \infty, f; G_\infty^{(l)}(r)) + m\left(r, \frac{f'}{f}\right) \\ &= m(r, \infty, f; G_\infty^{(l)}(r)) + O(\log r) \quad (r > r_0) \end{aligned}$$

and hence, we deduce from (3.3) and (5.3)

$$\bar{\delta}^{(l)}(\infty, f) \geq \delta^{(l)}(\infty, f') = 1/p \quad (l=1, 2, \dots, p).$$

As  $\sum_{l=1}^p \bar{\delta}^{(l)}(\infty, f) \leq \bar{\delta}(\infty, f) \leq \delta(\infty, f) = 1$ , we have

$$\delta(\infty, f) = \bar{\delta}(\infty, f) = \sum_{l=1}^p \bar{\delta}^{(l)}(\infty, f) = 1$$

and

$$\bar{\delta}^{(l)}(\infty, f) = 1/p \quad (l=1, 2, \dots, p).$$

As  $\bar{\delta}(\infty, f) + \underline{\delta}(\infty, f) \leq \delta(\infty, f)$ , we have

$$\underline{\delta}(\infty, f) = 0 \quad (\text{so that } \underline{\delta}^{(l)}(\infty, f) = 0 \quad (l=1, 2, \dots, p)).$$

Since we have

$$(5.4) \quad \log^+ \left| \frac{1}{f-a_k} \right| \leq \log^+ \left| \frac{1}{f'} \right| + \log^+ \left| \frac{f'}{f-a_k} \right|,$$

$$m(r, a_k, f; G_\infty(r)) \leq m(r, 0, f'; G_\infty(r)) + m\left(r, \frac{f'}{f-a_k}\right)$$

$$= O(\log r) \quad (r > r_0).$$

Hence

$$\bar{\Delta}(a_k, f) = 0 \quad (\text{so that } \bar{\delta}^{(l)}(a_k, f) = 0 \quad (l=1, 2, \dots, p)).$$

Therefore

$$\Delta(a_k, f) \leq \bar{\Delta}(a_k, f) + \underline{\Delta}(a_k, f) = \underline{\Delta}(a_k, f) \leq \Delta(a_k, f)$$

and hence, we deduce from Lemma A of [2] (p. 59)

$$\delta(a_k, f) = \Delta(a_k, f) = \underline{\Delta}(a_k, f).$$

By (5.4)

$$m(r, a_k, f; G_0^{(l)}(r)) \leq m(r, 0, f'; G_0^{(l)}(r)) + m\left(r, \frac{f'}{f-a_k}\right)$$

$$= m(r, 0, f'; G_0^{(l)}(r)) + O(\log r),$$

so that, in view of (3.3) and (5.2), we have

$$\bar{\delta}^{(l)}(a_k, f) \leq \delta^{(l)}(0, f') = 1/p \quad (l=1, 2, \dots, p).$$

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